Problem 5

For the following series, write formulas for the sequences a_n, S_n, and R_n, and find the limits of the sequences as $n \to \infty$ (if the limits exist).

$$\sum_{0}^{\infty} e^{2n \ln \sin(\pi/3)} \qquad \textit{Hint:} \text{ Simplify this.}$$

Solution

$$a_n = e^{2n \ln \sin(\pi/3)} = e^{2n \ln(\sqrt{3}/2)} = e^{\ln(\sqrt{3}/2)^{2n}} = \left(\frac{\sqrt{3}}{2}\right)^{2n} = \left(\frac{3}{4}\right)^n$$

$$S_n = \sum_{i=0}^{n} \left(\frac{3}{4}\right)^i = \frac{1 - \left(\frac{3}{4}\right)^{n+1}}{1 - \left(\frac{3}{4}\right)} = 4 - 3\left(\frac{3}{4}\right)^n$$

$$S = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \left[4 - 3\left(\frac{3}{4}\right)^n\right] = 4 - 3(0) = 4$$

$$R_n = S - S_n = 4 - \left[4 - 3\left(\frac{3}{4}\right)^n\right] = 3\left(\frac{3}{4}\right)^n$$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \left(\frac{3}{4}\right)^n = 0$$

$$\lim_{n\to\infty} R_n = \lim_{n\to\infty} 3\left(\frac{3}{4}\right)^n = 0$$